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Lie Algebras arising from Systems of Linear Differential Equations

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Abstract

It is known that (a) if the commutator of $A(t)$ and $A'(t)$ vanishes, or (b) if the commutator of $A(t_1)$ and $A(t_2)$ vanishes for all t_1 and t_2, then the solution of the initial-value problem for the system of linear ordinary differential equations $Y' = A(t)\,Y$ can be written as $Y = \exp[A(t)]$. Obviously condition (b) implies (a). In this report examples are investigated in which (a) is satisfied but not (b). This may happen even when $A(t)$ is a matrix of polynomials, and even when $A(t)$ cannot be transformed into triangular form. The present work extends results obtained in a previous paper by W. Magnus, Report No. BR-3.

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1. Introduction

Let $A(t)$ be $n \times n$ matrix whose elements depend analytically on a real parameter t. We consider the general system of n homogeneous linear differential equations of the first order for n unknown functions, with $A' = \frac{dA}{dt}$ as its matrix of coefficients. We may write this system in the form

$$\frac{dY}{dt} = A'Y \tag{1.1}$$

where Y is an $n \times n$ matrix whose column vectors represent a set of n linearly independent solutions of

$$\frac{dy_\nu}{dt} = \sum_{\mu=1}^{n} a'_{\nu,\mu}(t)\, y_\mu, \quad A' = (a'_{\nu,\mu}), \quad \nu = 1,2,\ldots, n. \tag{1.2}$$

Let I denote the unit matrix; then if A' is independent of t, the natural and elementary form of the general solution of (1.1) with the initial conditions

$$Y(0) = I \tag{1.3}$$

is

$$Y(t) = \exp[\Omega(t)] \tag{1.4}$$

with $\Omega(t) = tA'$. It is known[1] that a solution of type (1.4) always exists for sufficiently small values of t. Under certain circumstances, $\Omega(t)$ can be expressed in terms of a finite number of quadratures involving $A(t)$. The simplest case is $\Omega = \exp[A(t)]$ which is known to be a solution of (1.1) if the commutator

$$[A'(t), A(t)] = A'(t)\, A(t) - A(t)\, A'(t) \equiv 0. \tag{1.5}$$

The next case is

$$\Omega(t) = A(t) - \frac{1}{2}\int_0^t [A'(\tau), A(\tau)]\, d\tau \tag{1.6}$$

which gives a solution of (1.1) for all t if

$$(1.7) \qquad \left[A'(t), \int_0^t \left[A'(\tau), A(\tau)\right] d\tau\right] \equiv 0$$

and

$$(1.8) \qquad \left[\left[A'(t), A(t)\right] A(t)\right] \equiv 0 .$$

(See Magnus [1] for a discussion of this point.)

We shall confine ourselves to an investigation of the condition (1.5). It is trivially satisfied if

$$(1.9) \qquad \left[A(s), A(t)\right] \equiv 0$$

for all values of s and t. However, in general (1.9) is not a consequence of (1.5), and we shall show that (1.5) provides a non-trivial test for integrability in finite terms. The conditions (1.7) and (1.8) and the corresponding conditions for 'higher' cases could be treated in the same manner.

2. Commutator Relations

Let $A(t)$ be an $n \times n$ matrix which is analytic in t, that is let

$$(2.1) \qquad A(t) = \sum_{m=0}^{\infty} A_m t^m,$$

where the A_m are constant matrices. Then $\left[A(t), A'(t)\right] \equiv 0$ implies

$$(2.2) \qquad \sum_{r=0}^{[m/2]} (m-2r) \left[A_{m-r}, A_r\right] = 0, \qquad m = 1,2,3,\ldots .$$

We propose to prove that it follows from (2.1) that certain multiple commutators of A_o and any one of the A_m vanish. We shall use the abbreviation

$$\left\{B, A_o^{m-1}\right\} = \left[\left[\cdots\left[\left[B, A_o\right] A_o\right]\cdots\right] A_o\right]$$

for an (m-1)-fold commutator of A_o and B. Now we can state:

Lemma 1. For $m = 2,3, \ldots$;

$$\left\{A_m, A^{m-1}\right\} = 0 . \tag{2.3}$$

Proof. Since we have from (2.2)

$$\left[A_1, A_o\right] = \left[A_2, A_o\right] = 0 \tag{2.4}$$

we may use induction with respect to m in proving (2.3).

We observe that (2.2) allows us to express $\left[A_m, A_o\right]$ as a linear combination

$$\left[A_m, A_o\right] = \sum_{k=1}^{m-1} c_k \left[A_{m-k}, A_k\right] \tag{2.5}$$

of the commutators $\left[A_{m-k}, A_k\right]$. Now we apply repeated bracket multiplication (from the right) by A_o to (2.5) and use the Jacobi identity

$$\left[\left[A, B\right]C\right] + \left[\left[C, A\right]B\right] + \left[\left[B, C\right]A\right] = 0 \tag{2.6}$$

and the anti-commutativity relation

$$\left[A, B\right] = -\left[B, A\right] . \tag{2.7}$$

By a repeated application of (2.6) and (2.7) to

$$\left\{A_m, A_o^{m-1}\right\} = \sum_{k=1}^{m-1} c_k \left\{\left[A_{m-k}, A_k\right], A_o^{m-2}\right\} \tag{2.8}$$

(which is a consequence of (2.5)) we find that the right-hand side in (2.8) can be expressed as a linear combination of terms of the types

$$\left[\left\{A_{m-k}, A_0^{m-2}\right\}, A_k\right] \tag{2.9}$$

and

$$\left[\left\{A_r, A_0^{\rho}\right\}, \left\{A_k, A_0^{\sigma}\right\}\right], \tag{2.10}$$

where $r + \rho + k + \sigma = 2m - 2$ and where $r < m$, $k < m$, $r + k = m$. Therefore in (2.10) either $\rho \geq r - 1$ or $\sigma \geq k - 1$. Otherwise, we would have $r + \rho + k + \sigma < 2r + 2k - 2 < 2m - 2$. According to our hypothesis of induction every term in (2.10) vanishes and therefore so does the left-hand side of (2.8). The types (2.9) vanish trivially because of the induction hypothesis, and this completes the proof of Lemma 1.

3. Reduction to the Case of a Single Eigenvalue of A_0

As a consequence of Lemma 1, we have the following:

Theorem I. Let $A(t)$ be given by (2.1) and let $[A(t), A'(t)] \equiv 0$. Let λ_ρ $(\rho = 1, 2, \ldots, r)$ be the different eigenvalues of A_0 and let j_ρ denote their multiplicities. Then there exists an $n \times n$ matrix C with coefficients independent of t such that

$$B(t) = C^{-1} A(t)\, C \tag{3.1}$$

assumes the form

$$B(t) = \begin{pmatrix} \boxed{B_1} & 0 & \cdot & \cdot & 0 \\ 0 & \boxed{B_2} & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & \boxed{B_r} \end{pmatrix} \tag{3.2}$$

where all elements outside the boxes are zero for all values of t, and where the matrices within the square boxes have, for t = 0, one eigenvalue only. If in (1.1) we put $Y = CZ$, then Z satisfies

$$\frac{dZ}{dt} = B'(t)\, Z \tag{3.3}$$

and the original system of linear differential equations has been split into independent systems involving only j_ρ unknown functions.

In order to prove Theorem I, we need the following:

Definition: Let λ be an eigenvalue of the matrix M; then the generalized eigenspace S_λ of M belonging to λ is defined as the set of all vectors $\vec{x}$ for which

$$(M - \lambda I)^k \ \vec{x} = 0$$

for a sufficiently large value of k. With this definition, we shall derive Theorem I from

Lemma 2: The matrix $A(t)$ maps upon itself each generalized eigenvector space S_ρ ($\rho = 1,2,\dots, r$) of A_o, where S_ρ belongs to λ_ρ.

In fact, as a consequence of this, we may introduce by means of a transformation C a new system of basis vectors such that each basis vector belongs to one of the S_ρ. It follows from the Jordan canonical form for A_o that the S_ρ span the space of all vectos $\vec{x}$. Therefore, the transformation C transforms $A(t)$ into a matrix $B(t)$ of the type described in Theorem I.

In order to prove Lemma 2, we have to show that all of the matrices A_m leave each S_ρ invariant.

Let $\vec{x}$ belong to the eigenspace $S_o(\lambda = 0)$ of A_o. There is no loss of generality here, since λ can be subtracted from each element of the principal diagonal of A_o when put into the Jordan canonical form. It must be shown that for any $A_\nu = D$, $D\vec{x}$ belongs to S_o.

First, we use Lemma 1:

$$(3.4) \qquad E = \left\{A_\nu, A_o^{\nu-1}\right\} = 0 .$$

If we write $\nu - 1 = \ell$ and $A_\nu = D$, then

$$(3.5) \qquad E = DA_o^\ell - \binom{\ell}{1} A_o D A_o^{\ell-1} + \binom{\ell}{2} A_o^2 D A_o^{\ell-2} + \dots + (-1)^\ell A_o^\ell D.$$

Let $\vec{x}$ be such that $A_o^{r+1}\vec{x} = 0$, $A_o^r \vec{x} \neq 0$. First we shall prove Lemma 2 for $r = 0$. In that case

$$E\vec{x} = (-1)^\ell A_o^\ell (D\vec{x}) = 0.$$

Therefore $A_o^\ell (D\vec{x}) = 0$, and $D\vec{x}$ belongs to S_o. Now we use induction on r. Assume that the Lemma has been proved for all vectors $\vec{z}$ for which $A_o^r \vec{z} = 0$. That is, assume that $A_o^k(D\vec{z}) = 0$ for some k if $A_o^r \vec{z} = 0$. Note that vectors $A_o \vec{x} = \vec{z}_1$, $A_o^2 \vec{x} = \vec{z}_2, \ldots, A_o^\ell \vec{x} = \vec{z}_\ell$ are all z-vectors. Form

$$(3.6) \qquad A_o^k E \vec{x} = A_o^k D\vec{z}_\ell - \binom{\ell}{1} A_o^{k+1} D\vec{z}_{\ell-1} + \binom{\ell}{2} A_o^{k+2} D\vec{z}_{\ell-2} + \cdots + \cdots + (-1)^\ell A_o^{k+\ell} D\vec{x} = 0.$$

Clearly, all terms in (3.6) vanish according to the above assumption, except for the last term. Therefore,

$$(-1)^\ell A_o^{k+\ell} (D\vec{x}) = 0.$$

Since k is indeterminate, merely sufficiently large, the invariance of the generalized eigenspaces of A_o is proved.

4. Constructions Based on a Theorem of Birkhoff

We shall construct an infinite number of examples of matrices A(t) satisfying (1.5) but not (1.9) by using the following argument:

Suppose that A(t) is a polynomial in t of degree ℓ. Then the equations (2.2) reduce to a finite number of $2\ell-1$ equations for the A_ν. The A_ν themselves generate a Lie-algebra under bracket multiplication, and if $\left[A'(t), A(t)\right]$ vanishes identically but $\left[A(s), A(t)\right]$ is not zero for all values of s and t, then this Lie algebra must be such that the relations

$$(4.1) \qquad \sum_{r=0}^{[m/2]} (m-2r) \left[A_{m-r}, A_r\right] = 0 \quad (m = 1, 2, \ldots, 2\ell - 1)$$

hold but not all of the commutators $\left[A_{m-r}, A_r\right]$ vanish.

We shall now construct an abstract Lie-ring Λ related to the hypothetical Lie-ring generated by the matrices A_ν. Let $a_\nu (\nu = 0,1,\dots,\ell)$ be the generators of Λ and let Λ be defined by the relations

$$(4.2) \qquad \sum_{r=0}^{[m/2]} (m - 2r) \left[a_{m-r}, a_r\right] = 0, \quad (m = 1,2\dots, 2\ell-1)$$

and

$$(4.3) \qquad \left[\left[a_\nu, a_\mu\right] a_\rho\right] = 0 , \quad (\nu, \mu, \rho = 0,1,\dots,\ell).$$

Then Λ has a finite basis of at least $\ell + 1 + \ell(\ell + 1)/2 - (2\ell - 1) = (\ell^2 - \ell + 4)/2$ linearly independent elements, namely, the a_ν and a subset of linearly independent commutators $\left[a_\nu, a_\mu\right]$. Among the $\ell(\ell + 1)/2$ commutators, at least $\ell(\ell + 1)/2 - (2\ell - 1)$ must be linearly independent because there are only the $(2\ell - 1)$ linear relations (4.2) postulated for them. The relations (4.3) do not reduce the number of linearly independent commutators $\left[a_\nu, a_\mu\right]$ because all these relations are homogeneous with respect to the a_ν. The purpose in postulating the relations (4.3) was to insure the existence of a finite basis for Λ and to permit the application of Birkhoff's theorem[3].

According to this theorem of Birkhoff there exists a faithful representation of the nilpotent Lie-algebra Λ by finite matrices which are properly triangular. Any such representation (say, $a_\nu \rightarrow A_\nu$) is of the type we are investigating here, and then we have

Theorem II. There exist infinitely many finite matrices which are polynomials in t and which satisfy (1.5) but not (1.9).

This result answers a basic question raised by the work of Schwerdtfeger[2].

5. Examples

The Lie-algebras associated with the matrices $A(t)$ of Theorem II are all nilpotent. We consider the matric polynomial of degree three:

$$A(t) = A_o + A_1 t + A_2 t^2 + A_3 t^3 \tag{5.1}$$

where

$$A(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & t^3 & t \\ 1 & 0 & 0 & 0 \\ 3t^2 & 0 & 0 & 0 \end{pmatrix} .$$

(5.1) satisfies (1.5) but not (1.9), since

$$\left[A_1, A_2\right] = -3\left[A_o, A_3\right] \neq 0 .$$

It is interesting to note that any matric polynomial (5.1) of degree three generates a Lie-algebra which is 3-nilpotent. This result is readily established, because from (1.5) it follows that

$$\begin{gathered} \left[A_o, A_1\right] = \left[A_o, A_2\right] = \left[A_1, A_3\right] = \left[A_2, A_3\right] = 0 \\ \left[A_1, A_2\right] = -3\left[A_o, A_3\right] . \end{gathered} \tag{5.3}$$

Choosing $\left[A_1, A_2\right]$ as the linearly independent commutator and using bracket multiplication on the right by A_o, A_1, A_2, A_3 successively, each time applying the Jacobi identity, we have the desired result.

However, the Lie-algebra of the A_ν need not be nilpotent; in fact, it may not even be solvable. These facts are established by the following example due to Ascoli[4]:

$$A(t) = A_0 + A_1 t + A_2 t^2 + A_3 t^3 + A_4 t^4 . \tag{5.4}$$

where

$$A(t) = \begin{pmatrix} t^2 & t^3 & t^4 \\ -2t & -2t^2 & -2t^3 \\ 1 & t & t^2 \end{pmatrix} . \tag{5.5}$$

(5.5) satisfies (1.5) but not (1.9), since not all $[A_i, A_j] = 0$ $(i, j = 0,1,2,3,4, \ i \neq j)$. Actually, $[A_1, A_2] \neq 0$, $[A_1, A_3] \neq 0$, $[A_3, A_2] \neq 0$. Furthermore from (5.4) and (5.5) we obtain

$$\begin{aligned} \big[[A_1, A_2], [A_1, A_3]\big] &= 2\,[A_1, A_2] \\ \big[[A_1, A_2], [A_3, A_2]\big] &= -9\,[A_1, A_3] \\ \big[[A_1, A_3], [A_3, A_2]\big] &= 2\,[A_3, A_2] . \end{aligned} \tag{5.6}$$

From (5.6) it follows that the commutators of commutators of commutators, etc., here only serve to give constant multiples of $[A_1, A_2]$, $[A_1, A_3]$, and $[A_3, A_2]$, and hence never vanish. Therefore, these matrices do not form a solvable Lie-algebra. Also, the A_ν of this Ascoli example cannot form a nilpotent Lie-algebra since non-solvability implies non-nilpotency.

The Ascoli example shows that it is impossible to transform all matrices A_ν simultaneously into triangular form, for commutators of commutators of commutators, etc., must ultimately vanish for triangular matrices. But in the above case this is not so, since it has just been shown that the Lie-algebra generated is non-solvable.

Thus, the exponential solution of the system of linear differential equations

$$\frac{dY}{dt} = A'(t) \cdot Y \tag{5.7}$$

gives a solution in terms of a single quadrature which can not be obtained in a more direct way by stepwise integration of a triangular system.

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